Calibrated Submanifolds of $\mathbb{R}^7$ and $\mathbb{R}^8$ with Symmetries

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1 Introduction

In this article, we describe a method of constructing certain types of calibrated submanifold of $\mathbb{R}^7$ and $\mathbb{R}^8$ with *symmetries*. The main result is the exhibition of explicit examples of U(1)-invariant *associative* cones in $\mathbb{R}^7$ and *Cayley 4-folds* in $\mathbb{R}^8$ which are invariant under SU(2). This research is motivated by the work of Joyce in [4] on special Lagrangian (SL) m-folds in $\mathbb{C}^m$, and the work of the author in [6].

In Section 2, we describe the calibrations and calibrated submanifolds that are the focus of our study. These are called *associative 3-folds* and *coassociative 4-folds* in $\mathbb{R}^7$ and *Cayley 4-folds* in $\mathbb{R}^8$.

The method of construction to produce calibrated submanifolds with *symmetries* is discussed in Section 3. The key result is that we may define examples using a system of first-order ordinary differential equations. This section also reviews the relevant material from [6].

Sections 4 and 5 contain the explicit examples. The first gives the system of differential equations defining U(1)-invariant associative cones. These equations are solved in a special case to give a 4-dimensional family of associative cones over T^2. Further, using the material in [6, §6] and these cones, we produce examples of *ruled* associative 3-folds.

Section 5 considers Cayley 4-folds invariant under an action of SU(2). The family of all Cayley 4-folds invariant under this action is described using a real octic and three real quartics. Cayley 4-folds invariant under SU(2) are also considered in [1]; there is some overlap between our example and those given in this reference.

The final section gives some further examples of systems of ordinary dif-

ferential equations defining associative, coassociative and Cayley submanifolds, each associated with a symmetry group which is described.

Notes

(a) Manifolds are assumed to be nonsingular and submanifolds to be immersed unless stated otherwise.

(b) By a *cone* in $\mathbb{R}^n$ we shall mean a dilation-invariant submanifold of $\mathbb{R}^n$ which is nonsingular except possibly at 0.

2 Calibrated submanifolds of $\mathbb{R}^7$ and $\mathbb{R}^8$

2.1 Calibrated geometry

We define *calibrations* and *calibrated submanifolds* following the approach in [2].

Definition 2.1 Let (M, g) be a Riemannian manifold. An *oriented tangent k-plane V* on M is an oriented k-dimensional vector subspace V of T_xM, for some x in M. Given an oriented tangent k-plane V on M, $g|_V$ is a Euclidean metric on V and hence, using $g|_V$ and the orientation on V, there is a natural volume form, vol_V, which is a k-form on V.

A closed k-form η on M is a *calibration* on M if $\eta|_V \leq \mathrm{vol}_V$ for all oriented tangent k-planes V on M, where $\eta|_V = \kappa \cdot \mathrm{vol}_V$ for some $\kappa \in \mathbb{R}$, so $\eta|_V \leq \mathrm{vol}_V$ if $\kappa \leq 1$. An oriented k-dimensional submanifold N of M is a *calibrated submanifold* or *η-submanifold* if $\eta|_{T_xN} = \mathrm{vol}_{T_xN}$ for all $x \in N$.

Calibrated submanifolds are *minimal* submanifolds [2, Theorem II.4.2]. The minimality of calibrated submanifolds provides the following property, as discussed in [2].

Theorem 2.2 *A calibrated submanifold is real analytic wherever it is nonsingular.*

2.2 Associative and coassociative submanifolds of $\mathbb{R}^7$

The convention we adopt here for calibrations on $\mathbb{R}^7$ agree with [3, Chapter 10].

Definition 2.3 Let $(x_1, \ldots, x_7)$ be coordinates on $\mathbb{R}^7$ and write $d\mathbf{x}_{ij\ldots k}$ for the form $dx_i \wedge dx_j \wedge \ldots \wedge dx_k$. Define a 3-form φ_0 by:

$$\varphi_0 = d\mathbf{x}_{123} + d\mathbf{x}_{145} + d\mathbf{x}_{167} + d\mathbf{x}_{246} - d\mathbf{x}_{257} - d\mathbf{x}_{347} - d\mathbf{x}_{356}. \qquad (1)$$

By [2, Theorem IV.1.4], φ_0 is a calibration on $\mathbb{R}^7$ and submanifolds calibrated with respect to φ_0 are called *associative 3-folds*.

The 4-form $*\varphi_0$, where φ_0 and $*\varphi_0$ are related by the Hodge star, is given by:

$$* \varphi_0 = d\mathbf{x}_{4567} + d\mathbf{x}_{2367} + d\mathbf{x}_{2345} + d\mathbf{x}_{1357} - d\mathbf{x}_{1346} - d\mathbf{x}_{1256} - d\mathbf{x}_{1247}. \tag{2}$$

By [2, Theorem IV.1.16], $*\varphi_0$ is a calibration on $\mathbb{R}^7$, and $*\varphi_0$-submanifolds are called *coassociative 4-folds*.

Remark The form φ_0 is often referred to as the G_2 3-form on $\mathbb{R}^7$ since the Lie group G_2 may be defined as the stabilizer of φ_0 in $\mathrm{GL}(7,\mathbb{R})$.

We have a far more useful description of coassociative 4-folds which follows from [2, Proposition IV.4.5 & Theorem IV.4.6].

Proposition 2.4 *A 4-dimensional submanifold M of $\mathbb{R}^7$, with an appropriate orientation, is coassociative if and only if $\varphi_0|_M \equiv 0$.*

2.3 Cayley submanifolds of $\mathbb{R}^8$

Our definition of a distinguished 4-form on $\mathbb{R}^8$ used to describe Cayley 4-folds agrees with the convention in [3, Chapter 10].

Definition 2.5 Let $(x_1, \ldots, x_8)$ be coordinates on $\mathbb{R}^8$ and write $d\mathbf{x}_{ij\ldots k}$ for the form $dx_i \wedge dx_j \wedge \ldots \wedge dx_k$. Define a 4-form Φ_0 by:

$$\begin{aligned}\Phi_0 = {} & d\mathbf{x}_{1234} + d\mathbf{x}_{1256} + d\mathbf{x}_{1278} + d\mathbf{x}_{1357} - d\mathbf{x}_{1368} - d\mathbf{x}_{1458} - d\mathbf{x}_{1467} \\ & + d\mathbf{x}_{5678} + d\mathbf{x}_{3478} + d\mathbf{x}_{3456} + d\mathbf{x}_{2468} - d\mathbf{x}_{2457} - d\mathbf{x}_{2367} - d\mathbf{x}_{2358}.\end{aligned} \tag{3}$$

By [2, Theorem IV.1.24], Φ_0 is a calibration on $\mathbb{R}^8$, and submanifolds calibrated with respect to Φ_0 are called *Cayley 4-folds*.

Remark The stabilizer of Φ_0 in $\mathrm{GL}(8,\mathbb{R})$ is the Lie group $\mathrm{Spin}(7)$. We may thus refer to Φ_0 as the $\mathrm{Spin}(7)$ 4-form.

3 Constructing examples with symmetries

3.1 Evolution equations

In [6], an *evolution equation* for associative 3-folds in $\mathbb{R}^7$ was derived as a generalisation of the work of Joyce [4] on special Lagrangian m-folds in $\mathbb{C}^m$. The

proof relies on Theorem 2.2 and the following result from Harvey and Lawson [2, Theorem IV.4.1].

Theorem 3.1 *Let P be a 2-dimensional real analytic submanifold of $\mathbb{R}^7$. There locally exists a real analytic associative 3-fold N in $\mathbb{R}^7$ which contains P. Moreover, N is locally unique.*

We now present the theorem [6, Theorem 4.3].

Theorem 3.2 *Let P be a compact, orientable, 2-dimensional, real analytic manifold, χ a real analytic nowhere vanishing section of $\Lambda^2 TP$ and $\psi : P \to \mathbb{R}^7$ a real analytic embedding (immersion). There exist $\epsilon > 0$ and a unique family $\{\psi_t : t \in (-\epsilon, \epsilon)\}$ of real analytic maps $\psi_t : P \to \mathbb{R}^7$ with $\psi_0 = \psi$ satisfying*

$$\left(\frac{d\psi_t}{dt}\right)^d = (\psi_t)_*(\chi)^{ab}(\varphi_0)_{abc}(g_0)^{cd}, \tag{4}$$

where $(g_0)^{cd}$ is the inverse of the Euclidean metric on $\mathbb{R}^7$, using index notation for tensors on $\mathbb{R}^7$. Define $\Psi : (-\epsilon, \epsilon) \times P \to \mathbb{R}^7$ by $\Psi(t,p) = \psi_t(p)$. Then $M = \mathrm{Image}\,\Psi$ is a nonsingular embedded (immersed) associative 3-fold in $\mathbb{R}^7$.

We sketch the key ideas in the proof. Since P is compact and P, χ, ψ are real analytic, the *Cauchy–Kowalevsky Theorem* [7, Theorem B.1] from the theory of partial differential equations gives a family of maps ψ_t as stated. We may therefore define Ψ and M as in the statement of the theorem. Theorem 3.1 implies there locally exists a locally unique associative 3-fold N containing $\psi(P)$. Showing that N and M agree near $\psi(P)$, using the fact that φ_0 is a calibration, allows us to deduce that M is associative.

Using the associative case as a model we can quickly derive analogous evolution equations for coassociative and Cayley 4-folds.

We first require two results, [2, Theorem IV.4.3] and [2, Theorem IV.4.6], which are both similar to Theorem 3.1.

Theorem 3.3 *Suppose P is a 3-dimensional real analytic submanifold of $\mathbb{R}^7$ such that $\varphi_0|_P \equiv 0$. There locally exists a real analytic coassociative 4-fold N in $\mathbb{R}^7$ which contains P. Moreover, N is locally unique.*

Remark Unlike Theorem 3.1, we have to impose an extra condition on the boundary submanifold P in order to extend it to a coassociative 4-fold in $\mathbb{R}^7$.

Theorem 3.4 *Suppose P is a 3-dimensional real analytic submanifold of $\mathbb{R}^8$. There locally exists a real analytic Cayley 4-fold N in $\mathbb{R}^8$ which contains P. Moreover, N is locally unique.*

With these results at our disposal, it is clear that we may prove results like Theorem 3.2 for coassociative and Cayley 4-folds in exactly the same manner, so we omit the proofs.

Theorem 3.5 *Let P be a compact, orientable, 3-dimensional, real analytic manifold, χ a real analytic nowhere vanishing section of $\Lambda^3 TP$ and $\psi : P \to \mathbb{R}^7$ a real analytic embedding (immersion) such that $\psi^*(\varphi_0) \equiv 0$ on P. There exist $\epsilon > 0$ and a unique family $\{\psi_t : t \in (-\epsilon, \epsilon)\}$ of real analytic maps $\psi_t : P \to \mathbb{R}^7$ with $\psi_0 = \psi$ satisfying*

$$\left(\frac{d\psi_t}{dt}\right)^e = (\psi_t)_*(\chi)^{abc}(*\varphi_0)_{abcd}(g_0)^{de} \tag{5}$$

using index notation for tensors on $\mathbb{R}^7$, where $(g_0)^{de}$ is the inverse of the Euclidean metric on $\mathbb{R}^7$. Define $\Psi : (-\epsilon, \epsilon) \times P \to \mathbb{R}^7$ by $\Psi(t, p) = \psi_t(p)$. Then $M = \mathrm{Image}\,\Psi$ is a nonsingular embedded (immersed) coassociative 4-fold in $\mathbb{R}^7$.

Note The condition $\psi^*(\varphi_0)|_P \equiv 0$ implies that φ_0 vanishes on the real analytic 3-fold $\psi(P)$ in $\mathbb{R}^7$ and allows us to apply Theorem 3.3 as required.

Theorem 3.6 *Let P be a compact, orientable, 3-dimensional, real analytic manifold, χ a real analytic nowhere vanishing section of $\Lambda^3 TP$ and $\psi : P \to \mathbb{R}^8$ a real analytic embedding (immersion). There exist $\epsilon > 0$ and a unique family $\{\psi_t : t \in (-\epsilon, \epsilon)\}$ of real analytic maps $\psi_t : P \to \mathbb{R}^8$ with $\psi_0 = \psi$ satisfying*

$$\left(\frac{d\psi_t}{dt}\right)^e = (\psi_t)_*(\chi)^{abc}(\Phi_0)_{abcd}(g_0)^{de} \tag{6}$$

using index notation for tensors on $\mathbb{R}^8$, where $(g_0)^{de}$ is the inverse of the Euclidean metric on $\mathbb{R}^8$. Define $\Psi : (-\epsilon, \epsilon) \times P \to \mathbb{R}^8$ by $\Psi(t, p) = \psi_t(p)$. Then $M = \mathrm{Image}\,\Psi$ is a nonsingular embedded (immersed) Cayley 4-fold in $\mathbb{R}^8$.

3.2 The symmetries method

Now that we have a means of constructing calibrated submanifolds of $\mathbb{R}^7$ and $\mathbb{R}^8$, we shall consider the situation where the submanifold has a large symmetry group. The imposition of symmetry on the system reduces its complexity. This observation motivates our method of construction, which is a generalisation of the work of Joyce in [4].

We know from the remarks after Definitions 2.3 and 2.5 that it is natural to consider subgroups of $\mathrm{G}_2 \ltimes \mathbb{R}^7$ or $\mathrm{Spin}(7) \ltimes \mathbb{R}^8$ as symmetry groups for our calibrated submanifolds.

Let us consider, for example, the associative case. Suppose that G is a Lie subgroup of $\mathrm{G}_2 \ltimes \mathbb{R}^7$ which has a two-dimensional orbit $\mathcal{O} \subseteq \mathbb{R}^7$. Theorem 3.2 allows us to evolve each point in $\mathcal{O}$ transversely to the action of G and hence, hopefully, construct an associative 3-fold with symmetry group G.

Formally, take χ to be a nowhere vanishing section of $\Lambda^2 T\mathrm{G}$, which can easily be determined by finding a basis for the Lie algebra of G. Define $\psi : \mathrm{G} \to \mathcal{O} \subseteq \mathbb{R}^7$ to be an embedding given by

$$\psi(\gamma) = \gamma \cdot (x_1, \ldots, x_7)$$

for $\gamma \in \mathrm{G}$, where $(x_1, \ldots, x_7)$ is a point in $\mathcal{O}$ and $\gamma \cdot (x_1, \ldots, x_7)$ denotes the action of G on $\mathbb{R}^7$. Finally, for $t \in \mathbb{R}$, let $\psi_t : \mathrm{G} \to \mathbb{R}^7$ be given by

$$\psi_t(\gamma) = \gamma \cdot \big(x_1(t), \ldots, x_7(t)\big),$$

where $x_1(t), \ldots, x_7(t)$ are smooth real-valued functions of t with $x_j(0) = x_j$ for $j = 1, \ldots, 7$.

We may thus calculate either side of (4) and get a coupled system of seven first-order differential equations in seven variables dependent on t; that is, of the form

$$\frac{d}{dt}\big(x_1(t), \ldots, x_7(t)\big) = \Big(y_1\big(x_1(t), \ldots, x_7(t)\big), \ldots, y_7\big(x_1(t), \ldots, x_7(t)\big)\Big)$$

for functions $y_1, \ldots, y_7 : \mathbb{R}^7 \to \mathbb{R}$.

Remark $y_1, \ldots, y_7$ are quadratic functions of their arguments.

By Theorem 3.2, a unique solution to this system exists for $t \in (-\epsilon, \epsilon)$, for some $\epsilon > 0$. Moreover, if

$$M = \big\{\gamma \cdot \big(x_1(t), \ldots, x_7(t)\big) \,:\, \gamma \in \mathrm{G},\, t \in (-\epsilon, \epsilon)\big\},$$

it is an associative 3-fold in $\mathbb{R}^7$ which is clearly G-invariant.

For the coassociative case, we need to consider Lie subgroups G of $\mathrm{G}_2 \ltimes \mathbb{R}^7$ which have a 3-dimensional orbit $\mathcal{O}$. However, we also need to choose $\mathcal{O}$ so that $\varphi_0|_{\mathcal{O}} = 0$; i.e. we need $\psi : \mathrm{G} \to \mathcal{O}$ to be an embedding such that $\psi^*(\varphi_0) \equiv 0$ on G.

To construct Cayley examples with symmetries, we need to focus on Lie subgroups of $\mathrm{Spin}(7) \ltimes \mathbb{R}^8$ that have 3-dimensional orbits.

Remark If we write the system of differential equations defining coassociative or Cayley 4-folds with symmetries in the form

$$\frac{d\mathbf{x}}{dt} = \mathbf{y}(\mathbf{x}),$$

the components of $\mathbf{y}$ will be cubic functions of the variables in $\mathbf{x}$.

The author has looked at a variety of different subgroups and has derived systems of differential equations defining associative, coassociative and Cayley submanifolds. However, in the majority of situations the author has been unsuccessful in solving the system. In Sections 4 and 5 we present two important cases which we have been able to solve. Some additional scenarios where the author has had less fortune are discussed in Section 6.

4 U(1)-invariant associative cones

In this section, we consider associative 3-folds which are invariant both under an action of U(1) on the $\mathbb{C}^3$ component of $\mathbb{R}^7 \cong \mathbb{R} \oplus \mathbb{C}^3$ and under dilations.

Definition 4.1 Let $\mathbb{R}^+$ denote the group of positive real numbers under multiplication. The group action of $\mathbb{R}^+ \times \mathrm{U}(1)$ on $\mathbb{R}^7 \cong \mathbb{R} \oplus \mathbb{C}^3$ is given by, for some fixed $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$,

$$(x_1, z_1, z_2, z_3) \longmapsto (rx_1,\, re^{is\alpha_1}z_1,\, re^{is\alpha_2}z_2,\, re^{is\alpha_3}z_3) \qquad r > 0,\, s \in \mathbb{R}.$$

To ensure we have a U(1) action in G_2, we choose α_1, α_2, α_3 to be coprime integers satisfying $\alpha_1 + \alpha_2 + \alpha_3 = 0$.

Define smooth maps $\psi_t : \mathbb{R}^+ \times \mathrm{U}(1) \to \mathbb{R}^7$ by

$$\psi_t(r, e^{is}) = \big(rx_1(t),\, re^{is\alpha_1}z_1(t),\, re^{is\alpha_2}z_2(t),\, re^{is\alpha_3}z_3(t)\big), \tag{7}$$

where $x_1(t)$, $z_1(t) = x_2(t) + ix_3(t)$, $z_2(t) = x_4(t) + ix_5(t)$ and $z_3(t) = x_6(t) + ix_7(t)$ are smooth functions of t.

Using (7) we calculate the tangent vectors to the group action given in Definition 4.1:

$$\mathbf{u} = (\psi_t)_*\left(\frac{\partial}{\partial r}\right) = \sum_{j=1}^{7} x_j \frac{\partial}{\partial x_j} \text{ and}$$

$$\mathbf{v} = (\psi_t)_*\left(\frac{\partial}{\partial s}\right)$$

$$= \alpha_1\left(x_2\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_2}\right) + \alpha_2\left(x_4\frac{\partial}{\partial x_5} - x_5\frac{\partial}{\partial x_4}\right) + \alpha_3\left(x_6\frac{\partial}{\partial x_7} - x_7\frac{\partial}{\partial x_6}\right).$$

If we take $\chi = \frac{\partial}{\partial r} \wedge \frac{\partial}{\partial s}$, then $(\psi_t)_*(\chi) = \mathbf{u} \wedge \mathbf{v}$. We deduce that, writing $\mathbf{e}_j = \frac{\partial}{\partial x_j}$,

$$\begin{aligned}
\mathbf{u}^a \mathbf{v}^b (\varphi_0)_{abc} (g_0)^{cd} = {} & \big(\alpha_1(x_2^2 + x_3^2) + \alpha_2(x_4^2 + x_5^2) + \alpha_3(x_6^2 + x_7^2)\big)\mathbf{e}_1 \\
& + \big(- \alpha_1 x_1 x_2 + (\alpha_2 - \alpha_3)(x_4 x_7 + x_5 x_6)\big)\mathbf{e}_2 \\
& + \big(- \alpha_1 x_1 x_3 + (\alpha_2 - \alpha_3)(x_4 x_6 - x_5 x_7)\big)\mathbf{e}_3 \\
& + \big(- \alpha_2 x_1 x_4 + (\alpha_3 - \alpha_1)(x_2 x_7 + x_3 x_6)\big)\mathbf{e}_4 \\
& + \big(- \alpha_2 x_1 x_5 + (\alpha_3 - \alpha_1)(x_2 x_6 - x_3 x_7)\big)\mathbf{e}_5 \\
& + \big(- \alpha_3 x_1 x_6 + (\alpha_1 - \alpha_2)(x_2 x_5 + x_3 x_4)\big)\mathbf{e}_6 \\
& + \big(- \alpha_3 x_1 x_7 + (\alpha_1 - \alpha_2)(x_2 x_4 - x_3 x_5)\big)\mathbf{e}_7.
\end{aligned}$$

We also have that

$$\frac{d\psi_t}{dt} = \sum_{j=1}^{7} \frac{dx_j(t)}{dt}\, \mathbf{e}_j.$$

Equating both sides of (4) using the above formulae as described in §3.2, we obtain the following theorem.

Theorem 4.2 *Use the notation of Definition 4.1. Let* $\beta_1 = \alpha_2 - \alpha_3$, $\beta_2 = \alpha_3 - \alpha_1$ *and* $\beta_3 = \alpha_1 - \alpha_2$. *Let* $x_1(t)$ *be a smooth real-valued function of* t *and let* $z_1(t)$, $z_2(t)$, $z_3(t)$ *be smooth complex-valued functions of* t *such that*

$$\frac{dx_1}{dt} = \alpha_1 |z_1|^2 + \alpha_2 |z_2|^2 + \alpha_3 |z_3|^2, \tag{8}$$

$$\frac{dz_1}{dt} = -\alpha_1 x_1 z_1 + i\beta_1 \overline{z_2 z_3}, \tag{9}$$

$$\frac{dz_2}{dt} = -\alpha_2 x_1 z_2 + i\beta_2 \overline{z_3 z_1} \text{ and} \tag{10}$$

$$\frac{dz_3}{dt} = -\alpha_3 x_1 z_3 + i\beta_3 \overline{z_1 z_2}. \tag{11}$$

These equations have a solution for all $t \in \mathbb{R}$ *and the subset* M *of* $\mathbb{R} \oplus \mathbb{C}^3 \cong \mathbb{R}^7$ *defined by*

$$M = \big\{\big(r x_1(t),\, r e^{is\alpha_1} z_1(t),\, r e^{is\alpha_2} z_2(t),\, r e^{is\alpha_3} z_3(t)\big) : r \in \mathbb{R}^+,\ s, t \in \mathbb{R}\big\}$$

is an associative 3-fold in $\mathbb{R}^7$. *Moreover,* (8)-(11) *imply that* $x_1^2 + |z_1|^2 + |z_2|^2 + |z_3|^2$ *can be chosen to be* 1 *and that* $\mathrm{Re}(z_1 z_2 z_3) = A$, *where* A *is a real constant.*

Proof: Noting that $\beta_1 + \beta_2 + \beta_3 = 0$, we immediately see that $x_1^2 + |z_1|^2 + |z_2|^2 + |z_3|^3$ is a constant which we can take to be one. This is hardly surprising since

the associative 3-fold was constructed so as to be a cone. We also see from (9)-(11) that

$$\frac{d}{dt}(z_1z_2z_3) = i(\beta_1|z_2|^2|z_3|^2 + \beta_2|z_3|^2|z_1|^2 + \beta_3|z_1|^2|z_2|^2),$$

which is purely imaginary, and therefore $\mathrm{Re}(z_1z_2z_3) = A$ is a constant.

Notice that the functions x_1, z_1, z_2 and z_3 are bounded, hence their first derivatives are bounded by (8)-(11). Thus, all of the functions which determine the behaviour of the solutions to (8)-(11) are bounded, from which it follows that they have solutions for all $t \in \mathbb{R}$. $\square$

Writing $z_j(t) = r_j(t)e^{i\theta_j(t)}$ for $j = 1, 2, 3$ and $\theta = \theta_1 + \theta_2 + \theta_3$, (8)-(11) become

$$\frac{dx_1}{dt} = \alpha_1 r_1^2 + \alpha_2 r_2^2 + \alpha_3 r_3^2; \tag{12}$$

$$\frac{dr_1}{dt} = -\alpha_1 x_1 r_1 + \beta_1 r_2 r_3 \sin\theta; \tag{13}$$

$$\frac{dr_2}{dt} = -\alpha_2 x_1 r_2 + \beta_2 r_3 r_1 \sin\theta; \tag{14}$$

$$\frac{dr_3}{dt} = -\alpha_3 x_1 r_3 + \beta_3 r_1 r_2 \sin\theta; \text{ and} \tag{15}$$

$$r_j^2 \frac{d\theta_j}{dt} = \beta_j A \qquad \text{for } j = 1, 2, 3, \tag{16}$$

with the conditions

$$x_1^2 + r_1^2 + r_2^2 + r_3^2 = 1 \text{ and} \tag{17}$$

$$r_1 r_2 r_3 \cos\theta = A. \tag{18}$$

We notice that we are restricted in our choices of the real parameter A. The problem of maximising A^2, by (17) and (18), is equivalent to the problem of maximising $r_1^2r_2^2r_3^2$ subject to $r_1^2+r_2^2+r_3^2 = 1$. By direct calculation the solution is $r_1^2 = r_2^2 = r_3^2 = \frac{1}{3}$. Therefore $A \in \left[-\frac{1}{3\sqrt{3}}, \frac{1}{3\sqrt{3}}\right]$. We can restrict to $A \geq 0$ since changing the sign of A corresponds to reversing the sign of $\cos\theta$, so the addition of π to θ.

The case where $A = \frac{1}{3\sqrt{3}}$ is immediately soluble since this forces $r_1 = r_2 = r_3 = \frac{1}{\sqrt{3}}$, which implies $x_1 = 0$ by (17) and $\cos\theta = 1$ by (18), so we can take $\theta = 0$. Equations (16) become

$$\frac{1}{3}\frac{d\theta_j}{dt} = \frac{1}{3\sqrt{3}}\beta_j \qquad \text{for } j = 1, 2, 3,$$

which can easily be solved, along with the condition $\theta = 0$, to give:

$$\theta_j(t) = \frac{\beta_j}{\sqrt{3}}\, t + \gamma_j \qquad \text{for } j = 1, 2, 3,$$

where γ_1, γ_2, γ_3 are real constants which sum to zero. Then

$$M = \left\{\left(0,\, re^{i\phi_1},\, re^{i\phi_2},\, re^{i\phi_3}\right) : r > 0,\, \phi_1, \phi_2, \phi_3 \in \mathbb{R},\, \phi_1 + \phi_2 + \phi_3 = 0\right\},$$

which is a $\mathrm{U}(1)^2$-invariant special Lagrangian cone, as studied in [2, §III.3.A], embedded in $\mathbb{R}^7$ and is therefore in itself not an interesting object of study here. Any associative 3-fold constructed with $x_1 = 0$ will be at least a U(1)-invariant special Lagrangian cone and so we shall not consider this situation further. However, we know that M must be the limiting case of the family of associative 3-folds parameterised by A as it tends to $\frac{1}{3\sqrt{3}}$.

We may also solve the equations in the following special case.

Theorem 4.3 *Use the notation of Theorem 4.2. Suppose that $\alpha_2 = \alpha_3$. Then x_1, z_1, z_2 and z_3 may be chosen to satisfy $x_1^2 + |z_1|^2 + |z_2|^2 + |z_3|^2 = 1$ and $\operatorname{Im} z_1 = 0$. Moreover, they satisfy:*

$$\begin{aligned} \operatorname{Re}(z_1 z_2 z_3) = A; \qquad & |z_1|(x_1^2 + |z_1|^2 - 1) = B; \\ \operatorname{Re}(z_1(z_2^2 - z_3^2)) = C; \quad \textit{and} \quad & \operatorname{Im}(z_1(z_2^2 + z_3^2)) = D \end{aligned}$$

for some real constants A, B, C and D.

Proof: Since $\beta_1 = 0$, (16) implies that the argument of z_1 is constant. Using U(1) we can take it to be zero so that z_1 is real. Moreover, $\beta_1 = 0$ and (17) imply that x_1 and z_1 evolve amongst themselves and hence, using (8) and (9), we deduce that the real function $f = |z_1|(x_1^2 + |z_1|^2 - 1)$ is constant. Note that SU(2) acts on the (z_2, z_3)-plane. We are thus led to calculate

$$\begin{aligned} &\frac{d}{dt}\left(z_1(az_2 + bz_3)(-\bar{b}z_2 + \bar{a}z_3)\right) \\ &\qquad\qquad = -4i\beta_2|z_1|^2 \operatorname{Re}(a\bar{b}z_2z_3) + i\beta_2|z_1|^2(|a|^2 - |b|^2)(|z_3|^2 - |z_2|^2) \end{aligned}$$

for constants $a, b \in \mathbb{C}$, which is purely imaginary. Equating real parts for $(a, b) = (1, -1)$ and $(a, b) = (i, 1)$ leads to the final two conserved quantities in the statement of the theorem. □

In Theorem 4.3, we have six conditions on seven variables, which thus determine the solution to the system of differential equations (8)-(11) and hence the

associative cone constructed by Theorem 4.2 for $\alpha_2 = \alpha_3$. Moreover, we may construct a function $\pi : \mathbb{R} \oplus \mathbb{C}^3 \to \mathbb{R}^6$ by mapping (x_1, z_1, z_2, z_3) to the six real constant functions given in Theorem 4.3, which are defined by the initial values $\big(x_1(0), z_1(0), z_2(0), z_3(0)\big)$.

Sard's Theorem [5, p. 173] states that if $f : X \to Y$ is a smooth map between finite-dimensional manifolds, the set of $y \in Y$ with some $x \in f^{-1}(y)$ such that $df|_x : T_xX \to T_yY$ is *not* surjective is of measure zero in Y. Therefore, $f^{-1}(y)$ is a submanifold of X of dimension $\dim X - \dim Y$ for almost all $y \in Y$. Applying Sard's Theorem, generically the fibres of π will be 1-dimensional submanifolds of $\mathbb{R} \oplus \mathbb{C}^3 \cong \mathbb{R}^7$. Moreover, we know that these fibres are compact by the conditions in Theorem 4.3. Hence, the variables form loops in $\mathbb{R}^7$ for generic initial values; i.e. the solutions are *periodic* in t. We deduce the following result.

Theorem 4.4 *Use the notation of Theorem 4.2 and suppose that $\alpha_2 = \alpha_3$. For generic values of the functions x_1, z_1, z_2 and z_3 at $t = 0$, the associative 3-folds constructed by Theorem 4.2 are closed* $\mathrm{U}(1)$*-invariant cones over T^2 in $\mathbb{R}^7$.*

This family of cones is determined by four real parameters, whereas the corresponding SL family, as discussed in [4, §7], is parameterised by one rational variable. Therefore, these cones are generically not SL.

We may also apply the theory described in [6, §6] to the family of cones given in Theorem 4.4 to produce examples of *ruled* associative 3-folds which are *asymptotically conical*. We thus define the terms we require, noting that a cone C in $\mathbb{R}^7$ is said to be *two-sided* if $C = -C$.

Definition 4.5 Let M be a 3-dimensional submanifold of $\mathbb{R}^7$. A *ruling* of M is a pair (Σ, π), where Σ is a 2-dimensional manifold and $\pi : M \to \Sigma$ is a smooth map, such that for all $\sigma \in \Sigma$ there exist $\mathbf{v}_\sigma \in \mathcal{S}^6$, $\mathbf{w}_\sigma \in \mathbb{R}^7$ such that $\pi^{-1}(\sigma) = \{r\mathbf{v}_\sigma + \mathbf{w}_\sigma : r \in \mathbb{R}\}$. Then the triple (M, Σ, π) is a *ruled submanifold* of $\mathbb{R}^7$.

An *r-orientation* for a ruling (Σ, π) of M is a choice of orientation for the affine straight line $\pi^{-1}(\sigma)$ in $\mathbb{R}^7$, for each $\sigma \in \Sigma$, which varies smoothly with σ. A ruled submanifold with an r-orientation for the ruling is called an *r-oriented ruled submanifold*.

Let (M, Σ, π) be an r-oriented ruled submanifold. For each $\sigma \in \Sigma$, let $\phi(\sigma)$ be the unique unit vector in $\mathbb{R}^7$ parallel to $\pi^{-1}(\sigma)$ and in the positive direction with respect to the orientation on $\pi^{-1}(\sigma)$, given by the r-orientation. Then $\phi : \Sigma \to \mathcal{S}^6$ is a smooth map. Define $\psi : \Sigma \to \mathbb{R}^7$ such that, for all $\sigma \in \Sigma$, $\psi(\sigma)$ is the unique vector in $\pi^{-1}(\sigma)$ orthogonal to $\phi(\sigma)$. Then ψ is a smooth map

and we may write

$$M = \{r\phi(\sigma) + \psi(\sigma) : \sigma \in \Sigma,\, r \in \mathbb{R}\}. \tag{19}$$

Define the *asymptotic cone* M_0 of a ruled submanifold M by

$$M_0 = \{\mathbf{v} \in \mathbb{R}^7 : \mathbf{v} \text{ is parallel to } \pi^{-1}(\sigma) \text{ for some } \sigma \in \Sigma\}.$$

If M is also r-oriented, then

$$M_0 = \{r\phi(\sigma) : \sigma \in \Sigma,\, r \in \mathbb{R}\} \tag{20}$$

and is usually a 3-dimensional two-sided cone; that is, whenever ϕ is an immersion.

Definition 4.6 Let M_0 be a closed cone in $\mathbb{R}^7$ and let M be a closed nonsingular submanifold in $\mathbb{R}^7$. We say that M is *asymptotically conical to M_0 with rate α*, for some $\alpha < 1$, if there exists some constant $R > 0$, a compact subset K of M and a diffeomorphism $\Psi : M_0 \setminus \bar{B}_R \to M \setminus K$ such that

$$\left|\nabla^k\big(\Psi(\mathbf{x}) - \iota(\mathbf{x})\big)\right| = O\big(r^{\alpha-k}\big) \quad \text{for } k \in \mathbb{N} \text{ as } r \to \infty,$$

where $\bar{B}_R$ is the closed ball of radius R in $\mathbb{R}^7$ and $\iota : M_0 \to \mathbb{R}^7$ is the inclusion map. Here $|\,.\,|$ is calculated using the cone metric on $M_0 \setminus \bar{B}_R$, and ∇ is a combination of the Levi–Civita connection derived from the cone metric and the flat connection on $\mathbb{R}^n$, which acts as partial differentiation.

We now use the construction involving *holomorphic vector fields* given in [6, Proposition 6.8]

Theorem 4.7 *Use the notation of Theorem 4.2 and suppose that $\alpha_2 = \alpha_3 = -1$. Let M, as given in Theorem 4.2, be an associative cone over T^2, which, by Theorem 4.4, occurs for generic choices of $x_1(0)$, $z_1(0)$, $z_2(0)$ and $z_3(0)$. Let $u, v : \mathbb{R}^2 \to \mathbb{R}$ be functions satisfying the Cauchy–Riemann equations and let $M_0 = M \cup (-M) \cup \{0\}$. The subset $M_{u,v}$ of $\mathbb{R} \oplus \mathbb{C}^3$ given by*

$$\begin{aligned} M_{u,v} = \Big\{\Big(& rx_1(t) + v(s,t)\big(2|z_1(t)|^2 - |z_2(t)|^2 - |z_3(t)|^2\big), \\ & e^{2is}\big(r + 2iu(s,t) - 2v(s,t)x_1(t)\big)z_1(t), \\ & e^{-is}\Big(\big(r - iu(s,t) + v(s,t)x_1(t)\big)z_2(t) - 3iv(s,t)\overline{z_3 z_1}\,\Big), \\ & e^{-is}\Big(\big(r - iu(s,t) + v(s,t)x_1(t)\big)z_3(t) + 3iv(s,t)\overline{z_1 z_2}\,\Big)\Big) : r, s, t \in \mathbb{R}\Big\} \end{aligned}$$

is an r-oriented ruled associative 3-fold in $\mathbb{R}^7 \cong \mathbb{R} \oplus \mathbb{C}^3$. Moreover, $M_{u,v}$ is asymptotically conical to M_0 with rate -1 in the sense of Definition 4.6.

Proof: Define $\phi : \mathbb{R}^2 \to \mathbb{R} \oplus \mathbb{C}^3 \cong \mathbb{R}^7$ by

$$\phi(s,t) = (x_1(t),\, e^{2is}z_1(t),\, e^{-is}z_2(t),\, e^{-is}z_3(t)).$$

Since $x_1^2 + |z_1|^2 + |z_2|^2 + |z_3|^2 = 1$, ϕ maps into $\mathcal{S}^6$, and we can write M_0 in the form (20). Define a holomorphic vector field w using u and v as follows:

$$w = u(s,t)\frac{\partial}{\partial s} + v(s,t)\frac{\partial}{\partial t}\,.$$

Define $\psi = \mathcal{L}_w\phi$, where $\mathcal{L}_w$ denotes the Lie deriviative with respect to w, and define $M_{u,v}$ by (19) for these choices of ϕ and ψ. Calculation using equations (8)-(11) of Theorem 4.2 shows that $M_{u,v}$ can be written as stated in the theorem. Applying [6, Proposition 6.8 & Theorem 6.9], since M_0 is a cone over T^2, gives us the various properties of $M_{u,v}$ as claimed. $\square$

Remark Although M and hence M_0 is U(1)-invariant, $M_{u,v}$ will not be in general.

5 SU(2)-invariant Cayley 4-folds

We consider three different natural actions of SU(2) on $\mathbb{C}^4 \cong \mathbb{R}^8$ in Spin(7), though the first two only give trivial examples of Cayley 4-folds. The first is where SU(2) acts on $\mathbb{C}^4 \cong \mathbb{C}^2 \oplus \mathbb{C}^2$ in the usual manner upon one $\mathbb{C}^2$ and trivially upon the other. The construction using this action gives an affine $\mathbb{C}^2 \subseteq \mathbb{C}^4$ as the Cayley 4-fold. The second is where SU(2) acts on $\mathbb{C}^4 \cong \mathbb{C}^3 \oplus \mathbb{C}$ as SO(3) on $\mathbb{C}^3$ and trivially on $\mathbb{C}$. The construction then produces a complex surface in $\mathbb{C}^4$ as the Cayley 4-fold, which may be written as follows:

$$\big\{(z_1, z_2, z_3, z_4) : z_1^2 + z_2^2 + z_3^2 = A,\ z_4 = B\big\}\,, \text{ where } A, B \in \mathbb{C} \text{ are constants.}$$

We therefore turn our attention to the diagonal action of SU(2).

Definition 5.1 Let

$$X = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in \mathrm{SU}(2),$$

where $a, b \in \mathbb{C}$ such that $|a|^2 + |b|^2 = 1$. Then X acts on $(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 \cong \mathbb{R}^8$ as:

$$X \cdot (z_1, z_2, z_3, z_4) = (az_1 + bz_2, -\bar{b}z_1 + \bar{a}z_2, az_3 + bz_4, -\bar{b}z_3 + \bar{a}z_4).$$

Define smooth maps $\psi_t : \mathrm{SU}(2) \to \mathbb{C}^4 \cong \mathbb{R}^8$ by:

$$\psi_t(X) = X \cdot \big(z_1(t), z_2(t), z_3(t), z_4(t)\big),$$

where $z_1(t)$, $z_2(t)$, $z_3(t)$ and $z_4(t)$ are smooth functions of t.

Calculation shows that we may take the following three complex matrices as a basis for the Lie algebra of SU(2) acting in this way:

$$U_1 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}; \qquad U_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix};$$

$$\text{and} \quad U_3 = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix}.$$

If we let $\mathbf{u}_j = (\psi_t)_*(U_j)$ for $j = 1, 2, 3$,

$$\mathbf{u}_1 = i\left(z_1\frac{\partial}{\partial z_1} - \bar{z}_1\frac{\partial}{\partial \bar{z}_1} - z_2\frac{\partial}{\partial z_2} + \bar{z}_2\frac{\partial}{\partial \bar{z}_2} + z_3\frac{\partial}{\partial z_3} - \bar{z}_3\frac{\partial}{\partial \bar{z}_3} - z_4\frac{\partial}{\partial z_4} + \bar{z}_4\frac{\partial}{\partial \bar{z}_4}\right),$$

$$\mathbf{u}_2 = z_2\frac{\partial}{\partial z_1} + \bar{z}_2\frac{\partial}{\partial \bar{z}_1} - z_1\frac{\partial}{\partial z_2} - \bar{z}_1\frac{\partial}{\partial \bar{z}_2} + z_4\frac{\partial}{\partial z_3} + \bar{z}_4\frac{\partial}{\partial \bar{z}_3} - z_3\frac{\partial}{\partial z_4} - \bar{z}_3\frac{\partial}{\partial \bar{z}_4} \quad \text{and}$$

$$\mathbf{u}_3 = i\left(z_2\frac{\partial}{\partial z_1} - \bar{z}_2\frac{\partial}{\partial \bar{z}_1} + z_1\frac{\partial}{\partial z_2} - \bar{z}_1\frac{\partial}{\partial \bar{z}_2} + z_4\frac{\partial}{\partial z_3} - \bar{z}_4\frac{\partial}{\partial \bar{z}_3} + z_3\frac{\partial}{\partial z_4} - \bar{z}_3\frac{\partial}{\partial \bar{z}_4}\right).$$

Thus, if we take $\chi = U_1 \wedge U_2 \wedge U_3$, $(\psi_t)_*(\chi) = \mathbf{u}_1 \wedge \mathbf{u}_2 \wedge \mathbf{u}_3$. Using the equations above for $\mathbf{u}_j$ and the formula (3) for Φ_0, we may calculate the right-hand side of (6):

$$\begin{aligned}
&\mathbf{u}_1^a \mathbf{u}_2^b \mathbf{u}_3^c (\Phi_0)_{abcd}(g_0)^{de} \\
&\qquad = \big(z_1\left(|z_1|^2 + |z_2|^2 + |z_3|^2 - |z_4|^2\right) + 2(\overline{z_1 z_4 - z_2 z_3} + z_2 z_3)\bar{z}_4\big)\,\frac{\partial}{\partial z_1} \\
&\qquad + \big(\bar{z}_1\left(|z_1|^2 + |z_2|^2 + |z_3|^2 - |z_4|^2\right) + 2(z_1 z_4 - z_2 z_3 + \overline{z_2 z_3})z_4\big)\,\frac{\partial}{\partial \bar{z}_1} \\
&\qquad + \big(z_2\left(|z_1|^2 + |z_2|^2 - |z_3|^2 + |z_4|^2\right) - 2(\overline{z_1 z_4 - z_2 z_3} - z_1 z_4)\bar{z}_3\big)\,\frac{\partial}{\partial z_2} \\
&\qquad + \big(\bar{z}_2\left(|z_1|^2 + |z_2|^2 - |z_3|^2 + |z_4|^2\right) - 2(z_1 z_4 - z_2 z_3 - \overline{z_1 z_4})z_3\big)\,\frac{\partial}{\partial \bar{z}_2}
\end{aligned}$$

$$+\left(z_3\left(|z_1|^2-|z_2|^2+|z_3|^2+|z_4|^2\right)-2(\overline{z_1z_4-z_2z_3}-z_1z_4)\bar{z}_2\right)\frac{\partial}{\partial z_3}$$
$$+\left(\bar{z}_3\left(|z_1|^2-|z_2|^2+|z_3|^2+|z_4|^2\right)-2(z_1z_4-z_2z_3-\overline{z_1z_4})z_2\right)\frac{\partial}{\partial \bar{z}_3}$$
$$+\left(z_4\left(-|z_1|^2+|z_2|^2+|z_3|^2+|z_4|^2\right)+2(\overline{z_1z_4-z_2z_3}+z_2z_3)\bar{z}_1\right)\frac{\partial}{\partial z_4}$$
$$+\left(\bar{z}_4\left(-|z_1|^2+|z_2|^2+|z_3|^2+|z_4|^2\right)+2(z_1z_4-z_2z_3+\overline{z_2z_3})z_1\right)\frac{\partial}{\partial \bar{z}_4}.$$

Moreover,

$$\frac{d\psi_t}{dt}=\sum_{j=1}^{4}\frac{dz_j}{dt}\frac{\partial}{\partial z_j}+\sum_{j=1}^{4}\frac{d\bar{z}_j}{dt}\frac{\partial}{\partial \bar{z}_j}.$$

Equating both sides of (6) and using Theorem 3.6 gives the following result.

Theorem 5.2 *Let $z_1(t), z_2(t), z_3(t), z_4(t)$ be smooth complex-valued functions of t satisfying*

$$\frac{dz_1}{dt}=z_1\left(|z_1|^2+|z_2|^2+|z_3|^2-|z_4|^2\right)+2(\overline{z_1z_4-z_2z_3}+z_2z_3)\bar{z}_4, \tag{21}$$
$$\frac{dz_2}{dt}=z_2\left(|z_4|^2+|z_1|^2+|z_2|^2-|z_3|^2\right)-2(\overline{z_1z_4-z_2z_3}-z_1z_4)\bar{z}_3, \tag{22}$$
$$\frac{dz_3}{dt}=z_3\left(|z_3|^2+|z_4|^2+|z_1|^2-|z_2|^2\right)-2(\overline{z_1z_4-z_2z_3}-z_1z_4)\bar{z}_2 \text{ and} \tag{23}$$
$$\frac{dz_4}{dt}=z_4\left(|z_2|^2+|z_3|^2+|z_4|^2-|z_1|^2\right)+2(\overline{z_1z_4-z_2z_3}+z_2z_3)\bar{z}_1 \tag{24}$$

for all $t\in(-\epsilon,\epsilon)$, for some $\epsilon>0$. The subset M of $\mathbb{C}^4\cong\mathbb{R}^8$ defined by

$$M=\left\{X\cdot\left(z_1(t),z_2(t),z_3(t),z_4(t)\right)\,:\,t\in(-\epsilon,\epsilon),\,X\in\mathrm{SU}(2)\right\},$$

where the action of $\mathrm{SU}(2)$ on $\mathbb{C}^4$ is given in Definition 5.1, is a Cayley 4-fold in $\mathbb{R}^8$.

We are able to give an explicit description of the Cayley 4-folds constructed in Theorem 5.2. Let $u(t)$ be a real-valued function satisfying

$$\frac{du}{dt}=2(|z_1|^2+|z_2|^2+|z_3|^2+|z_4|^2)u. \tag{25}$$

We observe, using (21)-(24), that the following quadratics satisfy (25):

$$|z_1|^2-|z_2|^2+|z_3|^2-|z_4|^2; \qquad z_1\bar{z}_2+z_3\bar{z}_4;$$
$$\mathrm{Re}(z_1z_4-z_2z_3); \quad \text{and} \qquad z_1\bar{z}_3+z_2\bar{z}_4.$$

Hence, each of these quadratics is a constant multiple of u. The first two correspond to the moment maps of the SU(2) action and the latter two are SU(2)-invariant. The first two quadratics are not SU(2)-invariant, but

$$\begin{aligned} Q(z_1, z_2, z_3, z_4) &= (|z_1|^2 - |z_2|^2 + |z_3|^2 - |z_4|^2)^2 + 4|z_1\bar{z}_2 + z_3\bar{z}_4|^2 \\ &= (|z_1|^2 + |z_2|^2)^2 + (|z_3|^2 + |z_4|^2)^2 + 2|z_1\bar{z}_3 + z_2\bar{z}_4|^2 - 2|z_1 z_4 - z_2 z_3|^2 \end{aligned} \tag{26}$$

is SU(2)-invariant and is a constant multiple of u^2.

Using (21)-(24), we calculate

$$\frac{d}{dt}\operatorname{Im}(z_1 z_4 - z_2 z_3) = -2(|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2)\operatorname{Im}(z_1 z_4 - z_2 z_3).$$

Therefore, by (25), $\operatorname{Im}(z_1 z_4 - z_2 z_3)$ is a constant multiple of u^{-1} and is an SU(2)-invariant quadratic. We then state our result, which is immediate from our discussion above.

Theorem 5.3 *Let A, B, C and D be real constants. Let $M \subseteq \mathbb{C}^4 \cong \mathbb{R}^8$ be defined by*

$$M = \{X \cdot (z_1, z_2, z_3, z_4) : X \in \mathrm{SU}(2)\},$$

where the action of $X \in \mathrm{SU}(2)$ on $\mathbb{C}^4$ is given in Definition 5.1 and z_1, z_2, z_3, z_4 satisfy:

$$Q(z_1, z_2, z_3, z_4)\left(\operatorname{Im}(z_1 z_4 - z_2 z_3)\right)^2 = A; \tag{27}$$

$$\operatorname{Re}(z_1 z_4 - z_2 z_3)\operatorname{Im}(z_1 z_4 - z_2 z_3) = B; \tag{28}$$

$$\operatorname{Re}(z_1\bar{z}_3 + z_2\bar{z}_4)\operatorname{Im}(z_1 z_4 - z_2 z_3) = C; \textit{ and} \tag{29}$$

$$\operatorname{Im}(z_1\bar{z}_3 + z_2\bar{z}_4)\operatorname{Im}(z_1 z_4 - z_2 z_3) = D, \tag{30}$$

with $Q(z_1, z_2, z_3, z_4)$ given by (26)*. Then M is a Cayley 4-fold in $\mathbb{R}^8$.*

The set of conditions (27)-(30) on the complex functions z_1, z_2, z_3, z_4 consists of setting one real octic and three real quartics to be constant, which defines a 4-dimensional subset of $\mathbb{C}^4$. Hence, Theorem 5.3 completely describes the SU(2)-invariant Cayley 4-folds given by Theorem 5.2.

6 Further examples

In this final section we present an example of a symmetry group and its corresponding system of ordinary differential equations for each type of calibrated submanifold considered in this paper. These equations are derived using the

method introduced in §3.2. Since the calculations involved in this method have already been described in detail through the work of the previous two sections, we feel justified in our omission of the relevant calculations here.

Though the author has had little success in attempting to solve the systems in this section himself, it is hoped that their exposition will be useful to others.

6.1 Associative 3-folds invariant under a subgroup of $\mathbb{R} \times \mathrm{U}(1)^2$

We may decompose $\mathbb{R}^7 \cong \mathbb{R} \oplus \mathbb{C}^3$, and so the action of $\mathbb{R} \times \mathrm{U}(1)^2$ on $\mathbb{R}^7$ may be written as:

$$(x_1, z_1, z_2, z_3) \longmapsto (x_1 + c,\, e^{i\phi_1} z_1,\, e^{i\phi_2} z_2,\, e^{-i(\phi_1+\phi_2)} z_3), \quad c, \phi_1, \phi_2 \in \mathbb{R}. \tag{31}$$

However, we want a two-dimensional orbit, so we choose a two-dimensional subgroup of $\mathbb{R} \times \mathrm{U}(1)^2$.

Definition 6.1 Let λ, μ, ν be real numbers which are not all zero. Define G to be the subgroup of $\mathbb{R} \times \mathrm{U}(1)^2$ which acts as in (31) with the following imposed:

$$\lambda c + \mu \phi_1 + \nu \phi_2 = 0. \tag{32}$$

If $\mu = \nu = 0$, then G is $\mathrm{U}(1)^2$. Suppose $\mu\nu \neq 0$. If there exist coprime integers p and q such that $\mu p + \nu q = 0$, then G is $\mathbb{R} \times \mathrm{U}(1)$ and otherwise it is an $\mathbb{R}^2$ subgroup.

Using the method of §3.2 provides the following theorem.

Theorem 6.2 *Let $x_1(t)$ be a smooth real-valued function of t and let $z_1(t)$, $z_2(t)$, $z_3(t)$ be smooth complex-valued functions of t such that*

$$\frac{dx_1}{dt} = 0\,, \tag{33}$$

$$\frac{dz_1}{dt} = -\nu z_1 - \lambda \overline{z_2 z_3}\,, \tag{34}$$

$$\frac{dz_2}{dt} = \mu z_2 - \lambda \overline{z_3 z_1} \text{ and} \tag{35}$$

$$\frac{dz_3}{dt} = (\nu - \mu) z_3 - \lambda \overline{z_1 z_2}\,, \tag{36}$$

using the notation from Definition 6.1. There exists $\epsilon > 0$ such that these equations have a solution for $t \in (-\epsilon, \epsilon)$ and the subset M of $\mathbb{R} \oplus \mathbb{C}^3 \cong \mathbb{R}^7$

defined by

$$M = \big\{\big(x_1(t)+c,\, e^{i\phi_1}z_1(t),\, e^{i\phi_2}z_2(t),\, e^{-i(\phi_1+\phi_2)}z_3(t)\big) : \\ t\in(-\epsilon,\epsilon),\, (c, e^{i\phi_1}, e^{i\phi_2})\in \mathrm{G}\big\}$$

is an associative 3-fold in $\mathbb{R}^7$*. Moreover,* M *does not lie in* $\{x\}\times\mathbb{C}^3$ *for any* $x\in\mathbb{R}$*, as long as not both* μ *and* ν *are zero, and* (34)-(36) *imply that* $\mathrm{Im}(z_1z_2z_3) = A$*, where* A *is a real constant.*

Proof: We only need to prove the last sentence in the statement above. We deduce immediately from (33) that x_1 is constant in the direction transverse to the group action, though it *is* changing along the group action (as long as not both μ and ν are zero), which means that M does not lie in $\{x\}\times\mathbb{C}^3$ for any real constant x in this case. We also note from (34)-(36) that

$$\frac{d}{dt}(z_1z_2z_3) = -\lambda(|z_2|^2|z_3|^2+|z_3|^2|z_1|^2+|z_1|^2|z_2|^2),$$

which is real, therefore $\mathrm{Im}(z_1z_2z_3)$ is a real constant. □

There are two trivial cases which may be solved immediately.

Firstly, suppose $\lambda = 0$. This is not geometrically interesting since it implies that G contains all possible translations in the first coordinate. Solving (34)-(36) shows that

$$M = \mathbb{R}\times\Big\{\Big(A_1e^{i\phi_1-\nu t},\, A_2e^{i\phi_2+\mu t},\, A_3e^{-i(\phi_1+\phi_2)+(\nu-\mu)t}\Big) : \\ t\in\mathbb{R},\, \mu\phi_1+\nu\phi_2 = 0\Big\},$$

where $A_1,\ A_2,\ A_3$ are complex constants such that $\mathrm{Im}(A_1A_2A_3) = A$. The expression in brackets above defines a holomorphic curve in $\mathbb{C}^3$.

The other case is when $\mu=\nu=0$. This forces $c=0$ in G, so there is no translation action in G, which means that M will be an embedded $\mathrm{U}(1)^2$-invariant SL 3-fold as studied in [2, §III.3.A]:

$$M = \{(x_1,z_1,z_2,z_3)\in\mathbb{R}^7 : \, x_1 = x,\, \mathrm{Im}(z_1z_2z_3) = A, \\ |z_1|^2-|z_3|^2 = B,\, |z_2|^2-|z_3|^2 = C\}$$

for some $x, A, B, C\in\mathbb{R}$.

6.2 $\mathrm{U}(1)^2$-invariant coassociative cones

We consider coassociative 4-folds invariant both under the action of $\mathrm{U}(1)^2$ on the $\mathbb{C}^3$ component of $\mathbb{R}^7\cong\mathbb{R}\oplus\mathbb{C}^3$ and under dilations.

Definition 6.3 Let $\mathbb{R}^+$ denote the group of positive real numbers under multiplication. Define an action of $\mathbb{R}^+ \times \mathrm{U}(1)^2$ on $\mathbb{R}^7 \cong \mathbb{R} \oplus \mathbb{C}^3$ by

$$(x_1, z_1, z_2, z_3) \longmapsto (rx_1,\, re^{i\phi_1}z_1,\, re^{i\phi_2}z_2,\, re^{-i(\phi_1+\phi_2)}z_3), \quad r > 0,\, \phi_1, \phi_2 \in \mathbb{R}. \tag{37}$$

We again apply the method described in §3.2, though this time we must choose our orbit so that φ_0 vanishes on it. This constraint imposes the condition $\mathrm{Re}(z_1z_2z_3) = 0$. We thus have the following result.

Theorem 6.4 *Let $x_1(t)$ be a smooth real-valued function of t and let $z_1(t)$, $z_2(t)$, $z_3(t)$ be smooth complex-valued functions of t satisfying*

$$\frac{dx_1}{dt} = -3\mathrm{Im}(z_1z_2z_3), \tag{38}$$

$$\frac{dz_1}{dt} = z_1(|z_2|^2 - |z_3|^2) + ix_1\overline{z_2z_3}, \tag{39}$$

$$\frac{dz_2}{dt} = z_2(|z_3|^2 - |z_1|^2) + ix_1\overline{z_3z_1} \text{ and} \tag{40}$$

$$\frac{dz_3}{dt} = z_3(|z_1|^2 - |z_2|^2) + ix_1\overline{z_1z_2}, \tag{41}$$

along with the condition

$$\mathrm{Re}(z_1z_2z_3) = 0 \tag{42}$$

at $t = 0$. The subset M of $\mathbb{R} \oplus \mathbb{C}^3 \cong \mathbb{R}^7$ defined by

$$M = \Big\{\big(rx_1(t),\, re^{i\phi_1}z_1(t),\, re^{i\phi_2}z_2(t),\, re^{-i(\phi_1+\phi_2)}z_3(t)\big) \,:\, r > 0,\, \phi_1, \phi_2, t \in \mathbb{R}\Big\}$$

is a coassociative 4-fold in $\mathbb{R}^7$. Moreover, (42) holds for all $t \in \mathbb{R}$ and $x_1^2 + |z_1|^2 + |z_2|^2 + |z_3|^2$ is a constant which can be taken to be 1.

Proof: It is immediate from (38)-(41) that $x_1^2 + |z_1|^2 + |z_2|^2 + |z_3|^2$ is a constant which can be chosen to be 1 without loss of generality. We may also calculate

$$\frac{d}{dt}(z_1z_2z_3) = ix_1(|z_2|^2|z_3|^2 + |z_3|^2|z_1|^2 + |z_1|^2|z_2|^2)$$

using (38)-(41) and deduce that $\mathrm{Re}(z_1z_2z_3)$ is a constant which has to be zero since (42) holds at $t = 0$. Theorem 3.5 only gives us that solutions to (38)-(41) exist for $t \in (-\epsilon, \epsilon)$ for some $\epsilon > 0$, but solutions exist for all t, as argued in the proof of Theorem 4.2, since the functions involved are all bounded. □

6.3 $\mathrm{U}(1)^2$-invariant Cayley cones

We conclude by turning our attention to Cayley cones which are invariant under a $\mathrm{U}(1)^2$ subgroup of $\mathrm{U}(1)^4$.

Definition 6.5 Let $\mathrm{G} \subseteq \mathrm{U}(1)^4$ be defined by

$$\mathrm{G} = \big\{(e^{i\alpha_1},\, e^{i\alpha_2},\, e^{i\alpha_3},\, e^{i\alpha_4}) \,:\, \alpha_1, \alpha_2,\, \alpha_3, \alpha_4 \in \mathbb{R} \text{ satisfy} \\ \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0 \text{ and } a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + a_4\alpha_4 = 0\big\}$$

for coprime integers a_1, a_2, a_3, a_4 with $a_1 + a_2 + a_3 + a_4 = 0$ and $a_1 \leq a_2 \leq a_3 \leq a_4$. This acts on $\mathbb{C}^4 \cong \mathbb{R}^8$ in the obvious way as a $\mathrm{U}(1)^2$ subgroup of $\mathrm{U}(1)^4$.

Theorem 6.6 *Use the notation of Definition 6.5. Let $z_j(t)$ for $j = 1, 2, 3, 4$ be smooth complex-valued functions of t satisfying*

$$\frac{dz_1}{dt} = a_1\overline{z_2z_3z_4} + \frac{1}{2}\,z_1\big((a_4 - a_3)|z_2|^2 + (a_2 - a_4)|z_3|^2 + (a_3 - a_2)|z_4|^2\big), \quad (43)$$

$$\frac{dz_2}{dt} = a_2\overline{z_3z_4z_1} + \frac{1}{2}\,z_2\big((a_4 - a_1)|z_3|^2 + (a_1 - a_3)|z_4|^2 + (a_3 - a_4)|z_1|^2\big), \quad (44)$$

$$\frac{dz_3}{dt} = a_3\overline{z_4z_1z_2} + \frac{1}{2}\,z_3\big((a_2 - a_1)|z_4|^2 + (a_4 - a_2)|z_1|^2 + (a_1 - a_4)|z_2|^2\big) \textit{ and} \quad (45)$$

$$\frac{dz_4}{dt} = a_4\overline{z_1z_2z_3} + \frac{1}{2}\,z_4\big((a_2 - a_3)|z_1|^2 + (a_3 - a_1)|z_2|^2 + (a_1 - a_2)|z_3|^2\big). \quad (46)$$

The subset M of $\mathbb{C}^4 \cong \mathbb{R}^8$ given by

$$M = \big\{\big(re^{i\alpha_1}z_1(t),\, re^{i\alpha_2}z_2(t),\, re^{i\alpha_3}z_3(t),\, re^{i\alpha_4}z_4(t)\big) \,: \\ r > 0,\, (e^{i\alpha_1}, e^{i\alpha_2}, e^{i\alpha_3}, e^{i\alpha_4}) \in \mathrm{G},\, t \in \mathbb{R}\big\}$$

is a Cayley 4-fold in $\mathbb{R}^8$. Moreover, $|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2$ is a constant which can be taken to be 1 *and* $\mathrm{Im}(z_1z_2z_3z_4) = A$ *for some real constant A.*

Proof: It is clear from (43)-(46) that $|z_1|^2 + \ldots + |z_4|^2$ is a constant and that we can take this constant to be 1 without loss of generality. Furthermore,

$$\frac{d}{dt}\,(z_1z_2z_3z_4) = a_1|z_2z_3z_4|^2 + a_2|z_3z_4z_1|^2 + a_3|z_4z_1z_2|^2 + a_4|z_1z_2z_3|^2,$$

which is purely real. Therefore $\mathrm{Im}(z_1z_2z_3z_4) = A$ is constant. Theorem 3.6 only gives existence of solutions of $t \in (-\epsilon, \epsilon)$ for some $\epsilon > 0$. However, by the same argument as in the proof of Theorem 4.2, solutions exist for all $t \in \mathbb{R}$, using the boundedness of the functions involved. $\square$

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